Lectures on height zeta functions: At the confluence of algebraic geometry, algebraic number theory, and analysis

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Antoine Chambert-Loir

IRMAR, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France
Courriel : `antoine.chambert-loir@univ-rennes1.fr`

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1. Introduction

1.1. Diophantine equations and geometry

1.1.1. Diophantine equations. — Broadly speaking, *arithmetic* is the study of diophantine equations, that is, systems of polynomial equations with *integral* coefficients, with a special emphasis on their solutions in rational integers. Of course, there are numerous variants, the most obvious ones allowing to consider coefficients and solutions in the field of rational numbers, or in more general number fields, or even in more general fields, *e.g.*, finite fields.

The reader should be warned that, in this generality, we are constrained by the *undecidability theorem* of Matijasevič (1971): there is no general method, that is no algorithm, to decide whether or not any given polynomial system has solutions in rational integers. Any mathematician working on diophantine equations is therefore obliged to consider specific types of diophantine equations, in the hope that such an undecidability issues do not apply within the chosen families of equations.

1.1.2. Enters geometry. — At first, one is tempted to sort equations according to the degrees of the polynomials which intervene. However, this approach is much too crude, and during the XXth century, mathematicians were led to realize that there are profound relations between the given diophantine equation and the *geometry* of its solutions in real or complex numbers. This led to considerations of geometric invariants such as the *genus* of an algebraic curve, to the (essentially opposite) notions of a variety of general type and a Fano variety, to the notion of a rational variety, etc.

In this survey, we are interested in diophantine equations having infinitely many solutions. A natural way to describe this infinite set consists in sorting the solutions according to their size (as integers) and in studying the asymptotic behaviour of the number of solutions of size smaller than a growing bound.

1.1.3. The circle problem. — The classical *circle problem* in analytic number theory is to estimate the number of integer vectors $\mathbf{x} \in \mathbf{Z}^n$ such that $s(\mathbf{x}) \leqslant B$, when $B \to \infty$ and $s(\cdot)$ is an appropriate notion of a size of a vector in $\mathbf{R}^n$. When $s(\cdot) = \|\cdot\|$ is a norm, for example the euclidean norm, this amounts to counting the number of lattice points in a ball with center 0 and of radius B.

Since a ball is convex, the leading term is easily seen to equal the volume of this ball; in other words,

$$\operatorname{Card}\{\mathbf{x} \in \mathbf{Z}^n \,;\, \|\mathbf{x}\| \leqslant B\} \sim B^n V_1,$$

where V_1 is the volume of the unit ball in $\mathbf{R}^n$. The study of the error term, however, is of a much more delicate nature. For $n = 2$, there is an easy $\mathrm{O}(B)$-bound which only requires the Lipschitz property of the boundary of the unit ball; when the norm is euclidean, one can prove a $\mathrm{O}(B^{2/3})$ bound using the positivity of the curvature of the boundary; however, the conjectured $\mathrm{O}(B^{1/2})$-bound remains open.

1.2. Elementary preview of Manin's problem

1.2.1. Projective space. — As indicated above, the size of a solution of a diophantine equation can be thought of as a measure of its complexity; for a vector $\mathbf{x} \in \mathbf{Z}^n$, it is standard to define its size to be its euclidean norm.

However, in algebra and geometry, we are often led to consider rational functions and their poles, which inevitably bring us to "infinity". The appropriate context to define and study the corresponding sizes, or as we shall now say, *heights*, is that of *projective geometry*. Recall that for any field F, one defines the n-dimensional projective space $\mathbf{P}^n(F)$ on F to be the set of lines in the vector space F^{n+1}. In other words, this is the quotient set of the set $F^{n+1} \setminus \{0\}$ of nonzero vectors modulo the action of homotheties: a point in $\mathbf{P}^n(F)$ can be described by a nonzero family of $n+1$ *homogeneous coordinates* $[x_0 : \cdots : x_n]$, while we are free to multiply these coordinates by a common nonzero element of F, still describing the same point.

To stick to the current terminology of algebraic geometry, this set $\mathbf{P}^n(F)$ will be called the set of *F-rational points of the projective space* $\mathbf{P}^n$ and the latter will be referred to as the *scheme* $\mathbf{P}^n$.

1.2.2. Heights. — Consider the case where $F = \mathbf{Q}$, the field of rational numbers. Any point $\mathbf{x} \in \mathbf{P}^n(\mathbf{Q})$ can be represented by $n+1$ rational numbers, not all zero; however, if we multiply these homogeneous coordinates by a common denominator, we see that we may assume them to be integers; we then may divide them by their greatest common divisor and obtain a system of homogeneous coordinates $[x_0 : \cdots : x_n]$ made of $n+1$ coprime integers. At this point, only one choice is left to us, namely multiplying this system by -1.

Consequently, we may define the *exponential height* of $\mathbf{x}$ as $H(\mathbf{x}) = \max(|x_0|, \dots, |x_n|)$ and its *logarithmic height* as $h(\mathbf{x}) = \log H(\mathbf{x})$. (Observe the notation, popularized by Serge Lang: small "h" for logarithmic height, capital "H" for exponential height.)

1.2.3. The theorems of Northcott and Schanuel. — Northcott (1950) made a fundamental, albeit trivial, observation: for any real number B, there are only finitely many points $\mathbf{x} \in \mathbf{P}^n(\mathbf{Q})$ such that $H(\mathbf{x}) \leqslant B$. Indeed, this amounts to saying that there are only finitely many systems of coprime integers $[x_0 : \cdots : x_n]$ such that $|x_i| \leqslant B$ for all i, considered up to multiplication by ± 1. Concretely, there are at most $(2B+1)^{n+1}$ such systems of integers, let it be coprime, and modulo ± 1!

More precisely, Schanuel (1979) proved that the number $N(B)$ of such points satisfies

$$N(B) \sim \frac{2^n}{\zeta(n+1)} B^{n+1}$$

when $B \to \infty$, where $\zeta(n+1)$ is Riemann's zeta function evaluated at $n+1$. When, say, $n = 2$, this can be interpreted as follows: for $B \to \infty$, the probability that two integers from $[1, B]$ are coprime tends to $1/\zeta(2) = 6/\pi^2$, a result originally due to Dirichlet; see Theorem 332 in the classic book of Hardy & Wright (1979).

The remarkable appearance of this notorious function reveals that something profound is happening before our eyes, something that would certainly appeal to any arithmetician.

1.2.4. The problem of Batyrev–Manin. — More generally, we want to consider subsets of the projective space defined by polynomial equations. Evaluating a polynomial in $n+1$ variables at a system of homogeneous coordinates of a point $\mathbf{x} \in \mathbf{P}^n(F)$ does not lead to a well-defined number, since the result changes very much if we multiply the coordinates by a common nonzero factor λ. However, if our polynomial is assumed to be homogeneous of some degree, the result is multiplied by λ^d; at least we can consistently say whether the result is zero or not.

Let thus $X \subset \mathbf{P}^n$ be a *closed subscheme* of $\mathbf{P}^n$, namely the set of common zeroes of a family of homogeneous polynomials in $n+1$ variables. In other words, for any field F, $X(F)$ is the subset of $\mathbf{P}^n(F)$ consisting of points satisfying this family of equations. In fact, we will also consider the case of locally closed subschemes of $\mathbf{P}^n$, that is, subschemes of the form $X \setminus Y$, where Y is a closed subscheme of $\mathbf{P}^n$ contained in a closed subscheme X.

We now define $N_X(B)$ to be the number of points $\mathbf{x} \in X(\mathbf{Q})$ such that $H(\mathbf{x}) \leqslant B$. The problem of Batyrev & Manin (1990) is to understand the asymptotic behaviour of $N_X(B)$, for $B \to \infty$. In particular, one is interested in the two numbers:

$$\beta_X^- = \liminf_{B\to\infty} \frac{\log N_X(B)}{\log B}, \qquad \beta_X^+ = \limsup_{B\to\infty} \frac{\log N_X(B)}{\log B}.$$

Observe that $\beta_X^- \leqslant \beta_X^+$ and that, when $X = \mathbf{P}^n$, $\beta_X^- = \beta_X^+ = n+1$.

1.2.5. Heath-Brown's conjecture. — However, some problems in analytic number theory require uniform upper bounds, e.g., bounds depending only on the degree and dimension of X. Heath-Brown conjectured that for any real number $\varepsilon > 0$ and any integer $d > 0$, there exists a constant $C(d,\varepsilon)$ such that $N_X(B) \leqslant C(d,\varepsilon) B^{\dim X + \varepsilon}$, for any integral closed subscheme X of $\mathbf{P}^n$ of degree d and dimension $\dim X$. (Degree and dimensions are defined in such a way that when H is a general subspace of codimension $\dim X$, $(X \cap H)(\mathbf{C})$ is a finite set of cardinality $\dim X$, counted with multiplicities. Here, "integral" means that X is not a nontrivial union of two subschemes.)

As an example for the conjecture, let us consider the polynomial $F = x_0x_2 - x_1x_3$ and the subscheme $X = V(F)$ in $\mathbf{P}^3$ defined by F. Considering lines on X, it may be seen that $N_X(B) \gg B^2$.

In fact, Browning *et al* (2006) established that this conjecture is equivalent to the same statement with X assumed to be a hypersurface defined by a single homogeneous polynomial of degree d. Let us also mention that this conjecture is "almost" a theorem: by work of Pila and the same authors, the only remaining degrees are $3 \leqslant d \leqslant 5$; moreover, the case of smooth hypersurfaces has been completed in Browning & Heath-Brown (2006).

1.2.6. Geometric parameters. — Let us discuss the parameters at our disposal to describe the counting function $N_X(B)$ and the exponents β_X^- and β_X^+.

The most important of them come from geometry. First of all, there is the dimension, as was already seen in HEATH-BROWN's conjecture.

The dimension appears also in SCHANUEL's theorem (precisely, the dimension plus one), but this is not a good interpretation of the exponent. Namely, one of the insights of Batyrev & Manin (1990) was the interpretation of $n+1$ as describing the location of the anticanonical divisor of $\mathbf{P}^n$ with respect to the cone of effective divisors. In effect, the n-differential form

$$d(x_1/x_0) \wedge d(x_2/x_0) \wedge \cdots \wedge d(x_n/x_0)$$

has a pole of multiplicity $n+1$ along the hyperplane defined by the equation $x_0 = 0$. Note that the rank of the Néron–Severi group will also enter the final picture — this is the group of classes of divisors where we identiy two divisors if they give the same degree to any curve drawn on the variety.

Properties of a more arithmetic nature intervene as well: the classical Hasse principle and weak approximation, but also their refinements within the framework of the theory of *Brauer-Manin obstructions*. This explains the appearance of orders of various Galois cohomology groups in asymptotic formulas.

Were we to believe that the result only depends on the "abstract" scheme X, we would rapidly find ourselves in contradictions. Indeed, there are many $\mathbf{P}^1$ that can be viewed in $\mathbf{P}^2$. On lines, the number of points of bounded height has the same order of magnitude, but the leading coefficient can change; for example, for lines with equations, say $x_0 = 0$, or $x_0 + x_1 + x_2 = 0$,j one finds $\beta^{\pm} = 2$, but the leading coefficient in $N_X(B)$ is $2/\zeta(2)$ in the first case, and only one half of it in the second case. There are also "non-linear lines" in $\mathbf{P}^2$, namely curves which can be parametrized by rational functions of one variable. The equation $x_2x_0 - x_1^2$ defines such a curve X' for which $\beta^{\pm}_{X'} = 1$, reflecting its non-linearity. In other words, even if we want to think in terms of an abstract variety X, the obtained formulas force us to take into account the *embedding* of X in a projective space $\mathbf{P}^n$.

There are also more subtle geometric and arithmetic caveats, related not only to X, but to its closed subschemes; we will consider these later.

1.2.7. The height zeta function. — Seeking tools and applications of analysis, we introduce the *height zeta function*. This is nothing other than the generating Dirichlet series $Z_X(s) = \sum_{\mathbf{x}\in X(\mathbf{Q})} H(\mathbf{x})^{-s}$, where s is a complex parameter.

This series converges for $\mathrm{Re}(s)$ large enough; for example, SCHANUEL's theorem implies that it converges for $\mathrm{Re}(s) > n+1$ and that it defines a holomorphic function on the corresponding half-plane.

A first, elementary, analytic invariant related to Z_X is its *abscissa of convergence* β_X. Since this Dirichlet series has positive coefficients, a result of LANDAU implies the inequality

$$\beta_X^- \leqslant \beta_X \leqslant \beta_X^+.$$

This is a very crude form of Abelian/Tauberian theorem, which we will use later on, once we have obtained precise analytic information about Z_X. Our goal will be to establish an analytic continuation for Z_X to a meromorphic function on some larger domain of the complex plane. Detailed information on the poles of this continuation will then imply a precise asymptotic expansion for $N_X(B)$.

2. Heights

2.1. Heights over number fields

2.1.1. Absolute values on the field of rational numbers. — Let us recall that an absolute value $|\cdot|$ on a field F is a function from F to $\mathbf{R}_+$ satisfying the following properties:

– $|a| = 0$ if and only if $a = 0$ (non-degeneracy);
– $|a + b| \leqslant |a| + |b|$ for any $a, b \in F$ (triangular inequality);
– $|ab| = |a|\,|b|$ for any $a, b \in F$ (multiplicativity).

Of course, the usual modulus on the field of complex numbers satisfies these properties, hence is an absolute value on $\mathbf{C}$. It induces an absolute value on any of its subfields, in particular on $\mathbf{Q}$. It is called archimedean since for any $a \in \mathbf{Q}$ such that $|a| \neq 0$, and any $T > 0$, there exists an integer n such that $|na| > T$. We write $|\cdot|_\infty$ for this absolute value.

In fact, the field of rational numbers possesses many other absolute values, namely the p-adic absolute value, where p is any prime number. It is defined as follows: Any nonzero rational number a can be written as $p^m u/v$, where u and v are integers not divisible by p and m is a rational integer; the integer m depends only on a and we define $|a|_p = p^{-m}$; we also set $|0|_p = 0$. Using uniqueness of factorization of integers into prime numbers, it is an exercise to prove that $|\cdot|_p$ is an absolute value on $\mathbf{Q}$. In fact, not only does the triangular inequality hold, but a stronger form is actually true: the *ultrametric* inequality:

– $|a + b|_p \leqslant \max(|a|_p, |b|_p)$, for any $a, b \in \mathbf{Q}$.

2.1.2. Ostrowski's theorem. — The preceding list gives us essentially all absolute values on $\mathbf{Q}$. Indeed, let $|\cdot|$ be an absolute value on $\mathbf{Q}$. By Ostrowski*'s theorem*, $|\cdot|$ is one of the following (mutually exclusive) absolute values:

– the trivial absolute value $|\cdot|_0$, defined by $|a|_0 = 1$ if $a \neq 0$ and $|0|_0 = 0$;
– the standard archimedean absolute value $|\cdot|_\infty$ and its powers $|\cdot|_\infty^s$ for $0 < s \leqslant 1$;
– the p-adic absolute value $|\cdot|_p$ and its powers $|\cdot|_p^s$ for $0 < s < \infty$ and some prime number p.

2.1.3. Topologies and completions. — To each of these families corresponds some distance on $\mathbf{Q}$ (trivial, archimedean, p-adic), given by $d(a, b) = |a - b|$, hence some topology. The trivial absolute value defines the discrete topology on $\mathbf{Q}$, the archimedean absolute values the usual topology of $\mathbf{Q}$, as a subspace of the real numbers, and the p-adic absolute values the so-called p-adic topology.

The standard process of Cauchy sequences now constructs for any of these absolute values a complete field in which $\mathbf{Q}$ is dense and to which the absolute value uniquely extends. The obtained field is $\mathbf{Q}_\infty = \mathbf{R}$ when the absolute value is the archimedean one, and is written $\mathbf{Q}_p$ when the absolute value is p-adic.

2.1.4. Product formula. — A remarkable equality relates all of these absolute values, namely: for any nonzero rational number a, $\prod_{p\leqslant\infty} |a|_p = 1$. This is called the *product formula*. (We have written $p \leqslant \infty$ to indicate that the set of indices p ranges over the set of prime numbers to which we adjoin the symbol ∞.) Let us decompose $a = \pm \prod_p p^{n_p}$ as a product of a sign and of (positive or negative) prime powers. For any prime number p, we have $|a|_p = p^{-n_p}$, while $|a|_\infty = \prod_p p^{n_p}$; the product formula follows at once from these formulae.

From now on, we shall ignore the trivial absolute value.

2.1.5. Number fields. — Let F be a number field, that is, a finite extension of $\mathbf{Q}$. Let $|\cdot|_v$ be a (non-trivial) absolute value on F. Its restriction to $\mathbf{Q}$ is an absolute value on $\mathbf{Q}$, hence is given by a power of the p-adic absolute value (for $p \leqslant \infty$), as in Ostrowski's list. The completion furnishes a complete field F_v which is again a finite extension of $\mathbf{Q}_p$; in particular, we have a norm map $\mathrm{N}\colon F_v \rightarrow \mathbf{Q}_p$. We say that v is *normalized* if $|a|_v = |N(a)|_p$.

We write $\mathrm{Val}(F)$ for the set of (non-trivial) normalized valuations on F. As in the case of $\mathbf{Q}$, the product formula holds: for any nonzero $a \in F$,

$$\prod_{v\in\mathrm{Val}(F)} |a|_v = 1.$$

2.1.6. Heights on the projective space. — Using this machinery from algebraic number theory, we may extend the definition of the height of a point in $\mathbf{P}^n(\mathbf{Q})$ to points in $\mathbf{P}^n(F)$.

Let $\mathbf{x}$ be a point of $\mathbf{P}^n(F)$, given by a system of homogeneous coordinates $[x_0 : \ldots : x_n]$ in F, not all zero. We may define the (exponential) height of $\mathbf{x}$ as

$$H_F(\mathbf{x}) = \prod_{v\in\mathrm{Val}(F)} \max(|x_0|_v, |x_1|_v, \ldots, |x_n|_v)$$

since the right hand side does not depend on the choice of a specific system of homogeneous coordinates. Indeed, if we replace x_i par ax_i, for some nonzero element of F, the right hand side gets multiplied by

$$\prod_{v\in\mathrm{Val}(F)} |a|_v = 1.$$

For $F = \mathbf{Q}$, this definition coincides with the one previously given. Indeed, we may assume that the coordinates x_i are coprime integers. Then, for any prime number p, $|x_i|_p \leqslant 1$ (since the x_is are integers), and one of them is actually equal to 1 (since they are not all divisible by p). Consequently, the p-adic factor is equal to 1 and

$$H_{\mathbf{Q}}(\mathbf{x}) = \max(|x_0|_\infty, \ldots, |x_n|_\infty) = H(\mathbf{x}).$$

Using a bit more of algebraic number theory, one can show that for any finite extension F' of F, and any $\mathbf{x} \in \mathbf{P}^n(F)$,

$$H_{F'}(\mathbf{x}) = H_F(\mathbf{x})^d, \qquad \text{where } d = [F' : F].$$

2.2. Line bundles on varieties

2.2.1. The notion of a line bundle. — One of the main differences between classical and modern algebraic geometry is the change of emphasis from "subvarieties of a projective space" to "varieties which can be embed in a projective space". Given an abstract variety X, the data of an embedding is essentially described by line bundles, a notion we now have to explain.

Let X be an algebraic variety over some field F. A *line bundle* L on X can be thought of as a "family" of lines $L(x)$ (its *fibres*) parametrized by the points x of X. There is actually a subtle point that not only rational points $x \in X(F)$ must be considered, but also points in $X(F')$, for all extensions F' (finite or not) of the field F. (We will try to hide these kind of complications.)

A line bundle L has *sections*: over an open subset U of X, such a section s induces maps $x \mapsto s(x)$ for $x \in U$, where $s(x) \in L(x)$ for any x. A section can be multiplied by a regular function: if s is a section of L on U and f is a regular function on U, then there is a section fs corresponding to the assignement $x \mapsto f(x)s(x)$. At the end, the set $\Gamma(U, L)$ of sections of L on U, also written $L(U)$, has a natural structure of a *module* on the ring $\mathscr{O}_X(U)$ of regular functions on U.

Sections can also be glued together: if s and s' are sections of L over open subsets U and U' which coincide on $U \cap U'$, then there is a unique section t on $U \cap U'$ which induces s on U and s' on U'.

What ties all of these lines together is that for every point $x \in X$, there is an open neighbourhood U of x, a *frame* ε_U which is a section of L on U such that $\varepsilon_U(x) \neq 0$ for all $x \in U$ and such that any other section s of L on U can be uniquely written as $f\varepsilon_U$, where f is a regular function on U.

2.2.2. The canonical line bundle. — The sections of the *trivial line bundle* $\mathscr{O}_X$ on an open set U are the ring $\mathscr{O}_X(U)$ of regular functions on U.

If X is smooth and everywhere n-dimensional, it possesses a canonical line bundle ω_X, defined in such a way that its sections on an open subset U are precisely the module of n-differential forms, defined algebraically as the n-th exterior product of the $\mathscr{O}_X(U)$-module of Kähler differentials $\Omega^1_{\mathscr{O}_X(U)/F}$.

2.2.3. The Picard group. — There is a natural notion of morphism of line bundles. If $f\colon L \to M$ is such a morphism, it assigns to any section $s \in \Gamma(U, L)$ a section $f(s) \in \Gamma(U, M)$ so that the map $s \mapsto f(s)$ is a morphism of $\mathscr{O}_X(U)$-modules (*i.e.*, additive and compatible with multiplication by regular functions).

An isomorphism is a morphism f for which there is an "inverse morphism" $g\colon M \to L$ such that $g \circ f$ and $f \circ g$ are the identical morphisms of L and M respectively.

From two line bundles L and M on X, one can construct a third one, denoted $L \otimes M$, in such a way that if ε and η are frames of L and M on an open set U, then $\varepsilon \otimes \eta$ is a frame of $L \otimes M$ on U, with the obvious compatibilities suggested by the tensor product notation, namely

$$(f\varepsilon) \otimes \eta = \varepsilon \otimes (f\eta) = f(\varepsilon \otimes \eta),$$

for any regular function on U.

Any line bundle L has an "inverse" L^{-1} for the tensor product; if ε is a frame of L on U, then a frame of L^{-1} on U is ε^{-1}, again with the obvious compatibilities

$$(f\varepsilon)^{-1} = f^{-1}\varepsilon^{-1}$$

for any nonvanishing regular function f on U.

These two laws are in fact compatible with the notion of isomorphism. The set of isomorphism classes of line bundles on X is an Abelian group, which is called the *Picard group* and denoted $\operatorname{Pic}(X)$.

2.2.4. Functoriality. — Finally, if $u\colon X \to Y$ is a morphism of algebraic varieties and L is a line bundle on Y, there is a line bundle u^*L on X defined in such a way that the fibre of u^*L at a point $x \in X$ is the line $L(u(x))$; similarly, if ε is a frame of L on an open set U of Y, then $u^*\varepsilon$ is a frame of u^*L on $u^{-1}(U)$.

At the level of isomorphism classes, this induces a map $u^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ which is a morphism of Abelian group.

2.3. Line bundles and embeddings

2.3.1. Line bundles on $\mathbf{P}^n$. — As a scheme over a field F, $\mathbf{P}^n$ parametrizes hyperplanes of the fixed vector space $V = F^{n+1}$. (In the introduction, we identified a point $\mathbf{x} \in \mathbf{P}^n(F)$ with the *line* in V generated by any system of homogeneous coordinates; this switch of point of view can be restored by duality.) In particular, to a point $x \in \mathbf{P}^n(F)$ corresponds a hyperplane $H(x)$ in V. Considering the quotient vectorspace $V/H(x)$, we hence get a line $L(x)$.

These lines $L(x)$ form a line bundle on $\mathbf{P}^n$. This line bundles admits sections $s_0, \dots, s_n$ defined on $\mathbf{P}^n$ which correspond to the homogeneous coordinates. Observe however that for $x \in \mathbf{P}^n(F)$, $s_i(x)$ is not a number, but a member of some line. However, given any generator $\varepsilon(x)$, of this line, there are elements $x_i \in F$ such that $s_i(x) = x_i\varepsilon(x)$ and the family $[x_0 : \dots : x_n]$ gives homogeneous coordinates for x.

It is useful to remember the relation $x_j s_i(x) = x_i s_j(x)$, valid for any $x \in \mathbf{P}^n(F)$ with homogeneous coordinates $[x_0 : \ldots : x_n]$ and any couple (i, j) of indices. It will also be necessary to observe that for any $x \in \mathbf{P}^n$, at least one of the $s_i(x)$ is nonzero — this is a reformulation of the fact that the homogeneous coordinates of a point are not all zero.

In traditional notation, this line bundle is denoted $\mathscr{O}_{\mathbf{P}^n}(1)$. Its class in the Picard group of $\mathbf{P}^n$ is a generator of this group, which is isomorphic to $\mathbf{Z}$. Similarly, the traditional notation for the line bundle corresponding to an integer $a \in \mathbf{Z}$ is $\mathscr{O}_{\mathbf{P}^n}(a)$.

The canonical line bundle of $\mathbf{P}^n$ is isomorphic to $\mathscr{O}_{\mathbf{P}^n}(-n-1)$. In fact, in the open subset U_i of $\mathbf{P}^n$ where the homogeneous coordinate x_i is nonzero, x_j/x_i defines a regular function and one has a differential form

$$\omega_i = d(x_0/x_i) \wedge d(x_1/x_i) \wedge \cdots \wedge \widehat{d(x_i/x_i)} \wedge \cdots \wedge d(x_n/x_i),$$

where the hat indicates that one omits the corresponding factor.

On the intersections $U_i \cap U_j$, one can check that the expressions $s_i(x)^{\otimes n+1}\omega_i$ and $s_j(x)^{\otimes n+1}\omega_j$ identify the one to the other if one uses the relation $x_i s_j(x) = x_j s_i(x)$

that we observed. This implies that these expressions can be glued together as a section of $\mathscr{O}_{\mathbf{P}^n}(n+1) \otimes \omega_{\mathbf{P}^n}$ which vanishes nowhere, thereby establishing the announced isomorphism.

2.3.2. Morphisms to a projective space. — Now we translate into the language of line bundles the geometric data induced by a morphism f from a variety X to a projective space $\mathbf{P}^n$. As we have seen, $\mathbf{P}^n$ is given with its line bundle $\mathscr{O}_{\mathbf{P}^n}(1)$ and its sections $s_0, \dots, s_n$ which do not vanish simultaneously. By functoriality, the morphism f furnishes a line bundle $f^*\mathscr{O}_{\mathbf{P}^n}(1)$ on X together with $(n+1)$ sections $f^*s_0, \dots, f^*s_n$ which, again, do not vanish simultaneously.

A fundamental fact in projective geometry is that this assignment defines a bijection between:

– the set of morphisms $f\colon X \to \mathbf{P}^n$;
– the set of data $(L, u_0, \dots, u_n)$ consisting of a line bundle L on X together with $n+1$ sections $u_0, \dots, u_n$ which do not vanish simultaneously.

We have only described one direction of this bijection, the other can be explained as follows. For $i \in \{0, \dots, n\}$, let U_i be the open subset of X where $u_i \neq 0$; by assumption, $U_0 \cup \dots \cup U_n = X$. On U_i, u_i is a frame of L and there are regular functions $f_{i0}, \dots, f_{in}$ on U_i such that, on that open set, $u_j = f_{ij} u_i$. This allows to define a morphism of algebraic varieties $f_i\colon U_i \to \mathbf{P}^n$ by the formula $x \mapsto [f_{i0}(x) : \dots : f_{in}(x)]$. It is easy to check that for any couple (i, j), the morphisms f_i and f_j agree on $U_i \cap U_j$, hence define a morphism $f\colon X \to \mathbf{P}^n$.

A simple example is given by the line bundle $\mathscr{O}_{\mathbf{P}^1}(d)$ on $\mathbf{P}^1$ and the sections $s_0^{\otimes i_0} \otimes s_1^{i_1}$, for $i_0 + i_1 = d$. It corresponds to the Veronese embedding of $\mathbf{P}^1$ in $\mathbf{P}^d$ defined by $[x : y] \mapsto [x^d : x^{d-1}y : \dots : xy^{d-1} : y^d]$.

2.3.3. Cones in the Picard group. — Let X be an algebraic variety and $\operatorname{Pic}(X)_{\mathbf{R}}$ the real vector space obtained by tensoring the group $\operatorname{Pic}(X)$ with the field of real numbers. Although we shall only study cases where it is finite-dimensional, this vector space may very well be infinite-dimensional.

The vector space $\operatorname{Pic}(X)_{\mathbf{R}}$ contains two distinguished cone. The first, Λ_{eff}, is called the *effective cone*, it is generated by line bundles which have nonzero sections over X. The second, Λ_{ample}, is called the *ample cone* and it is generated by line bundles of the form $f^*\mathscr{O}_{\mathbf{P}^n}(1)$, where f is an *embedding* of X into a projective space $\mathbf{P}^n$. (Such line bundles are called *very ample*, a line bundle which has a very ample power is called *ample*.)

One has $\Lambda_{\text{ample}} \subset \Lambda_{\text{eff}}$, because a very ample line bundle has nonzero sections, but the inclusion is generally strict.

For $X = \mathbf{P}^n$ itself, one has $\operatorname{Pic}(X) \simeq \mathbf{Z}$, hence $\operatorname{Pic}(X)_{\mathbf{R}} \simeq \mathbf{R}$, the line bundle $\mathscr{O}_{\mathbf{P}^n}(1)$ corresponding to the number 1, and both cones are equal to $\mathbf{R}_+$ under this identification.

2.4. Metrized line bundles

2.4.1. Rewriting the formula for the height. — Since an abstract variety may be embedded in many ways in a projective space, there are as many possible definitions for a height function on it. To be able to explain what happens, we first will rewrite the definition of the height of a point $\mathbf{x} \in \mathbf{P}^n(F)$ (where F is a number field) using line bundles.

Recall that $\mathrm{Val}(F)$ is the set of normalized absolute values on the number field F. Let $\mathbf{x} \in \mathbf{P}^n(F)$ and let $[x_0 : \ldots : x_n]$ be a system of homogeneous coordinates for $\mathbf{x}$. If $x_i \neq 0$, we may write, for any absolute value $v \in \mathrm{Val}(F)$,

$$\max(|x_0|_v, \ldots, |x_n|_v) = \left(\frac{|x_i|_v}{\max(|x_0|_v, \ldots, |x_n|_v)}\right)^{-1} |x_i|_v .$$

Recalling that s_i is the section of $\mathscr{O}_{\mathbf{P}^n}(1)$ corresponding to the ith homogeneous coordinate, we define

$$\|s_i(\mathbf{x})\|_v = \frac{|x_i|_v}{\max(|x_0|_v, \ldots, |x_n|_v)}.$$

We observe that the right hand side does not depend on the choice of homogeneous coordinates. Moreover, the product formula implies that

$$H_F(\mathbf{x}) = \prod_{v \in \mathrm{Val}(F)} \|s_i(\mathbf{x})\|_v^{-1} |x_i|_v = \prod_{v \in \mathrm{Val}(F)} \|s_i(\mathbf{x})\|_v^{-1} .$$

In other words, we have given a formula for $H_F(\mathbf{x})$ as a product over $\mathrm{Val}(F)$ where each factor is well defined, independently of any choice of homogeneous coordinates.

The reader should not rush to conclusions: so far we have only exchanged the indeterminacy of homogeneous coordinates with the choice of a specific index i, more precisely of a specific section s_i.

2.4.2. Metrized line bundles: an example. — The notation introduced for $\|s_i(\mathbf{x})\|$ suggests that it is the *v-adic norm* of the vector $s_i(\mathbf{x})$ in the fibre of $\mathscr{O}(1)$ at $\mathbf{x}$. A v-adic norm on a F-vector space E is a map $\|\cdot\|_v : E \to \mathbf{R}_+$ satisfying the following relations, analogous to those that the v-adic absolute value possesses:

- $\|e\|_v = 0$ if and only if $e = 0$ (non-degeneracy);
- $\|e + e'\|_v \leqslant \|e\|_v + \|e'\|_v$ for any $e, e' \in E$ (triangular inequality);
- $\|ae\|_v = |a|_v \|e\|_v$ for any $e \in E$ and any $a \in F$ (homogeneity).

In our case, the vector space is the line $E = \mathscr{O}(1)(\mathbf{x})$ and the norm of a single nonzero vector in E determines the norm of any other. Consequently, the formula given for $\|s_i(\mathbf{x})\|_v$ (which is a positive real number) uniquely extends to a norm on E. The formula $x_j s_i(\mathbf{x}) = x_i s_j(\mathbf{x})$ implies that the given norm does not depend on the initial choice of an index i such that $s_i(\mathbf{x}) \neq 0$.

Moreover, when the point $\mathbf{x}$ varies, the so-defined norms vary continuously in the sense that the norm of a section s on an open set U induces a continuous function from $U(F)$ (endowed with the v-adic topology) to $\mathbf{R}_+$. It suffices to check this fact on open sets which cover $\mathbf{P}^n$ and over which $\mathscr{O}(1)$ has frames; on the set where

$x_i \neq 0$, s_i is such a frame and the claim follows by observing that the given formula is continuous on $U_i(F_v)$.

2.4.3. Metrized line bundles: definition. — We now extend the previous construction to a general definition. Let X be an algebraic variety over a number field F and let L be a line bundle on X. A *v-adic metric* on L is the data of v-adic norms on the F_v-lines $L(x)$, when $x \in X(F_v)$, which vary continuously with the point x. This assertion means that for any open set U and any section s of L on U, the function $U(F_v) \to \mathbf{R}_+$ given by $x \mapsto \|s(x)\|_v$ is continuous. Since the absolute value of a regular function is continuous, it suffices to check this fact for frames whose open sets of definition cover X. A particular type of metrics is important; we call them *smooth metrics*. These are the metrics such that the norm of a local frame is $\mathscr{C}^\infty$ if v is archimedean, and locally constant if v is ultrametric.

The construction we have given in the preceding Section therefore defines a v-adic metric on the line bundle $\mathscr{O}_{\mathbf{P}^n}(1)$ on $\mathbf{P}^n$. This metric is smooth if v is ultrametric, but not if v is archimedean, because the function $(x, y) \mapsto \max(|x|, |y|)$ from $\mathbf{R}^2$ to $\mathbf{R}_+$ is not smooth. A variant of the construction furnishes a smooth metric in that case, namely the Fubini-Study metric, defined by replacing $\max(|x_0|_v, \dots, |x_n|_v)$ by $(|x_0|_v^2 + \dots + |x_n|_v^2)^{1/2}$.

2.4.4. Adelic metrics. — The formula for the height features all normalized absolute values of the number field F. We thus define an *adelic metric* on a line bundle L to be a family of v-adic metrics on L, for all $v \in \mathrm{Val}(F)$. However, to be able to define a height using such data, we need to impose a compatibility condition of "adelic type" on all these metrics.

Let us describe this condition. Let U be an affine open subset of X and ε be a frame of L on U. Let us represent U as a subvariety of some affine space $\mathbf{A}^N$ defined by polynomials with coefficients in F.

For any ultrametric absolute value $v \in \mathrm{Val}(F)$, the subset $\mathfrak{o}_v \subset F_v$ consisting of elements $a \in F_v$ such that $|a|_v \leqslant 1$ is therefore a subring of F_v. We may thus consider the subset $U(\mathfrak{o}_v) = U(F_v) \cap \mathfrak{o}_v^N$ of $U(F_v)$. Although it depends on the specific choice of a representation of U as a subvariety of $\mathbf{A}^N$, one can prove that two representations will define the same subsets $U(\mathfrak{o}_v)$ up to finitely many exceptions in $\mathrm{Val}(F)$. As a consequence, for any $x \in U(F)$, one has $x \in U(\mathfrak{o}_v)$ up to finitely many exceptions v (choose a representation where x has coordinates $(0, \dots, 0)$).

The *adelic compatibility condition* can now be expressed by requiring that for all $v \in \mathrm{Val}(F)$, up to finitely many exceptions, $\|\varepsilon(x)\|_v = 1$ for any $x \in U(\mathfrak{o}_v)$.

2.4.5. Heights for adelically metrized line bundles. — Let X be a variety over a number field F and L a line bundle on X with an adelic metric. For any $x \in X(F)$ and any frame s on a neighbourhood U of x, let us define

$$H_{L,s}(x) = \prod_{v \in \mathrm{Val}(F)} \|s(x)\|_v^{-1}.$$

By definition of an adelic metric, almost all of the terms are equal to 1. If t is any nonzero element of $L(x)$, there exists $a \in F^*$ such that $t = as(x)$; then $\|t\|_v = |a|_v \|s(x)\|_v$ for any $v \in \mathrm{Val}(F)$. In particular, $\|t\|_v = 1$ for almost all $v \in \mathrm{Val}(F)$ and

$$\prod_{v \in \mathrm{Val}(F)} \|t\|_v^{-1} = \prod_{v \in \mathrm{Val}(F)} \|as(x)\|_v^{-1} = \prod_{v \in \mathrm{Val}(F)} |a|_v^{-1} \|s(x)\|_v^{-1}$$
$$= \left(\prod_{v \in \mathrm{Val}(F)} |a|_v^{-1} \right) \left(\prod_{v \in \mathrm{Val}(F)} \|s(x)\|_v^{-1} \right) = \prod_{v \in \mathrm{Val}(F)} \|s(x)\|_v^{-1} = H_{L,s}(x)$$

where we used the product formula to establish the penultimate equality.

As a consequence, one can use any nonzero element of $L(x)$ in the formula $H_{L,s}(x)$; the resulting product does not depend on the choice of s. We write it $H_L(x)$.

For $L = \mathscr{O}_{\mathbf{P}^n}(1)$ with the adelic metric constructed previously, we recover the first definition of the height.

2.4.6. Properties of metrized line bundles. — There is a natural notion of tensor product of adelically metrized line bundles, for which the v-adic norm of a tensor product $e \otimes e'$ (for $e \in L(x)$ and $e' \in L'(x)$) is equal to $\|e\|_v \|e'\|_v$. With that definition, one has

$$H_{L \otimes L'}(x) = H_L(x) H_{L'}(x).$$

An isometry of adelically metrized line bundles is an isomorphism which preserves the metrics. The set of isometry classes of metrized line bundles form a group, called the *Arakelov-Picard group*, and denoted $\widehat{\mathrm{Pic}}(X)$. The above formula implies that the function from $X(F) \times \widehat{\mathrm{Pic}}(X)$ to $\mathbf{R}_+^*$ is linear in the second variable.

Let us moreover assume that X is projective. A v-adic metric on the the trivial line bundle is characterized by the norm of its section 1, which is a nonvanishing continuous functions ρ_v on $X(F_v)$. Consequently, an adelic metric ρ on $\mathscr{O}_X$ is given by a family (ρ_v) of such functions which are almost all equal to 1. (The projectivity assumption on X allows to write X as a finite union of open subsets U such that $X(F_v)$ is the union of the sets $U(\mathfrak{o}_v)$.) One has

$$H_\rho(x) = \prod_{v \in \mathrm{Val}(F)} \rho_v(x)^{-1}$$

(finite product). If ρ_v is identically equal to 1, we set $c_v = 1$. Otherwise, by continuity and compactness of $X(F_v)$, there is a positive real number c_v such that $c_v^{-1} < \rho_v(x) < c_v$ for any $x \in X(F_v)$. Let c be the product of all c_v (finite product); it follows that $c^{-1} < H_\rho(x) < c$ for any $x \in X(F)$.

As a consequence, if L and L' are two adelically metrized line bundles with the same underlying line bundle, the quotient of the exponential height functions H_L and $H_{L'}$ is bounded above and below (by a positive real number). This property is best expressed using the logarithmic heights $h_L = \log H_L$ and $h_{L'} = \log H_{L'}$: it asserts that the difference $h_L - h_{L'}$ is bounded.

2.4.7. Functoriality. — If $f\colon X\to Y$ is a morphism of algebraic varieties and L is a adelically metrized line bundle on Y, the line bundle f^*L on X has a natural adelic metric: indeed, the fibre of f^*L at a point x is the line $L(f(x))$, for which we use its given norm. This construction is compatible with isometry classes and defines a morphism of abelian groups $f^*\colon \widehat{\operatorname{Pic}}(Y)\to\widehat{\operatorname{Pic}}(X)$. At the level of heights, it implies the equality

$$H_L(f(x)) = H_{f^*L}(x)$$

for any $x\in X(F)$.

2.4.8. Finiteness property. — We have explained that for any real number B, $\mathbf{P}^n(\mathbf{Q})$ has only finitely many points of height smaller than B. As was first observed by Northcott (1950), this property extends to our more general setting: *if L is an adelically metrized line bundle whose underlying line bundle is ample, then the set of points $x\in X(F)$ such that $H_L(x)\leqslant B$ is finite.*

This can be proved geometrically on the basis of the result for the projective space over $\mathbf{Q}$. For simplicity of notation, we switch to logarithmic heights. Since L is ample, it is known that there exists an integer $d\geqslant 1$ such that $L^{\otimes d}$ is very ample, hence of the form $\varphi^*\mathscr{O}(1)$ for some embedding φ of X into a projective space $\mathbf{P}^n$. The functoriality property of heights together with the boundedness of the height when the underlying line bundle is trivial, imply that there is a positive real number c such that

$$h_L(x) = \frac{1}{d}h_{L^{\otimes d}}(x) \geqslant \frac{1}{d}h(\varphi(x)) + c.$$

Consequently, it suffices to show the finitess result on $\mathbf{P}^n$. Moreover, the explicit formula given for the height on $\mathbf{P}^n$ shows that

$$h([x_0:\cdots:x_n]) \geqslant \sup_{i,j} h([x_i:x_j]),$$

where the supremum runs over all couples (i,j) such that x_i and x_j are not both equal to 0. This reduces us to proving the finitess assertion on $\mathbf{P}^1$.

We also may assume that F is a Galois extension of $\mathbf{Q}$ and let $G=\{\sigma_1,\dots,\sigma_d\}$ be its Galois group. Let $x\in F$ and let us consider the polynomial

$$P_x = \prod_{\sigma\in G}(T-\sigma(x)) = T^d + a_1T^{d-1}+\cdots+a_d;$$

it has coefficients in $\mathbf{Q}$. For $v\in\operatorname{Val}(F)$ and $\sigma\in G$, the function $t\mapsto\left|\sigma(t)\right|_v$ is a normalized absolute value on F which is associated to the p-adic absolute value if so is v. Consequently,

$$\max(1,\left|\sigma_1(x)\right|_v,\dots,\left|\sigma_d(x)\right|_v) \leqslant \prod_{w|p}\max(1,\left|x\right|_w)$$

hence $h([1:\sigma_1(x):\ldots:|\sigma_d(x)|)\leqslant dh([1:x])$. Moreover, the map which associates to a point $[u_0:\ldots:u_d]\in\mathbf{P}^d$ the coefficients $[v_0:\ldots:v_d]$ of the polynomial

$\prod_{i=1}^{d}(u_0T - u_i)$ has degree d. This implies that

$$h([v_0 : \ldots : v_d]) = dh([u_0 : \ldots : u_d]) + \mathrm{O}(1).$$

If $h(x) \leqslant T$, this implies that the point $\mathbf{a} = [1 : a_1 : \ldots : a_d]$ of $\mathbf{P}^d(\mathbf{Q})$ is of height $h(\mathbf{a}) \leqslant d^2T$. By the finiteness property over $\mathbf{Q}$, there are only finitely many possible polynomials P_x, hence finitely many x since x is one of the d roots of P_x.

2.4.9. Rational points vs *algebraic points.* — To keep the exposition at the simplest level, we have only considered the height of F-rational points. However, there is a suitable notion of adelic metrics which uses not only the v-adic points $X(F_v)$, but all points of X over the algebraic closure $\overline{F_v}$ of F_v, or even its completion $\mathbf{C}_v$. Together with the obvious extension to metrized line bundles of the relation between $H_F(\mathbf{x})$ and $H_{F'}(\mathbf{x})$ when F' is a finite extension of F and $\mathbf{x} \in \mathbf{P}^n(F)$ this allows to define a (exponential) height function H_L on the whole of $X(\overline{F})$, as well as its logarithmic counterpart $h_L = \log H_L$. The preceding properties extend to this more general setting, the only nonobvious assertion being the boundedness of the height when the underlying line bundle is trivial.

3. Manin's problem

3.1. Counting functions and zeta functions

3.1.1. The counting problem. — Let X be a variety over a number field F and let L be an ample line bundle with an adelic metric. We have seen that for any real number B, the set of points $x \in X(F)$ such that $H_L(x) \leqslant B$ is a finite set. Let $N_X(L; B)$ be its cardinality. We are interested in the asymptotic behaviour of $N_X(L; B)$, when B grows to infinity. We are also interested in understanding the dependence on L.

3.1.2. Introducing a generating series. — As is common practice in all counting problems, *e.g.*, in combinatorics, one introduces a generating series. For the present situation, this is a *Dirichlet series*, called the *height zeta function*, and defined by

$$Z_X(L; s) = \sum_{x \in X(F)} H_L(x)^{-s},$$

for any complex number s such that the series converges absolutely. In principle, and we will eventually do so, one can omit the ampleness condition in that definition, but then the convergence in some half-plane is not assured (and might actually fail).

3.1.3. Abscissa of convergence. — The abscissa of convergence $\beta_X(L)$ is the infimum of all real numbers a such that $Z_X(L; s)$ converges absolutely for $\mathrm{Re}(s) > a$. Since the height zeta function is a Dirichlet series with positive coefficients, a theorem of Landau implies that $\beta_X(L)$ is also the infimum of all real numbers a such that $Z_X(L; a)$ converges.

Let us assume again that L is ample. Then, the proof of NORTHCOTT's theorem shows that $N_X(L; B)$ grows at most polynomially. It follows that $Z_X(L; s)$ converges for $\mathrm{Re}(s)$ large enough. For instance, let us assume that $N_X(L;B) \ll B^a$. Let us fix a real number $B > 1$. There are $\ll B^{na}$ points $x \in X(F)$ such that $H_L(x) \leqslant B^n$ and the part of the series given by points of heights between B^{n-1} and B^n is bounded by $B^{n(a-s)s}$. By comparison with the geometric series, we see that $Z_X(L;s)$ converges for $\mathrm{Re}(s) > a$.

This shows that the function $L \mapsto \beta_X(L)$ is well defined for adelically metrized line bundles whose underlying line bundle is ample. Moreover, it does not depend on the choice of an adelic metric, so it really comes from a function on the set of ample line bundles in $\mathrm{Pic}(X)$. The formula $H_{L^{\otimes d}}(x) = H_L(x)^d$ implies that $\beta_X(L^{\otimes d}) = d\beta_X(L)$ for any positive integer d. In other words, β_X is homogeneous of degree 1. One can also prove that it uniquely extends to a continuous homogeneous function on Λ_{ample} (see Batyrev & Manin (1990)).

3.1.4. Analytic properties of $Z_X(L;\cdot)$ vs. *asymptotic expansions of $N_X(L;\cdot)$.* — Despite this simple definition, very little is known about β_X, let it be about the height zeta function itself. However, there are numerous examples where $Z_X(L;\cdot)$ has a meromorphic continuation to some half-plane, with a unique pole of largest real value, say $\alpha_X(L)$, of order $t_X(L)$. By IKEHARA's Tauberian theorem, this implies an asymptotic expansion of the form

$$N_L(B) \sim cB^{\alpha_X(L)}(\log B)^{t_X(L)}.$$

In some cases, one can even establish terms of lower order, or prove explicit error terms.

However, these zeta functions are not as well behaved as the ones traditionally studied in algebraic number theory. It happens quite often, for example, that they have an essential boundary (Batyrev & Tschinkel (1995) show that this is already the case for some toric surfaces).

3.2. The largest pole

3.2.1. The influence of the canonical line bundle. — We now assume that X is smooth. One of the insights of mathematicians in the XXth century (notably MORDELL, LANG and VOJTA) was that the potential *density* of rational points is strongly related by the negativity of the canonical line bundle ω_X with respect to the ample cone.

This is quite explicit in the case of curves. Namely, if X is a curve of genus g, three cases are possible:

(1) **genus $g = 0$, ω_X^{-1} ample.** Then X is a *conic*. Two subcases are possible: either $X(F)$ is empty—X has no rational point— or X is isomorphic to the projective line. Moreover, the HASSE principle allows to decide quite effectively in which case we are.

(2) **genus $g = 1$, ω_X trivial.** If X has a rational point, then X is an *elliptic curve*, endowing $X(F)$ with a structure of abelian group. Moreover, the theorem of MORDELL-WEIL asserts that $X(F)$ is of finite type.

(3) **genus $g \geqslant 2$, ω_X ample.** By MORDELL's conjecture, first proved by Faltings (1983), $X(F)$ is a finite set.

The counting function distinguishes very clearly the first two cases. When X is the projective line, $N_X(B)$ grows like a polynomial in B (depending on how the height is defined), and for X an elliptic curve, $N_X(B) \approx (\log B)^{r/2}$, where r is the rank of the abelian group $X(F)$.

3.2.2. Increasing the base field. — The general conjectures of LANG lead us to expect that if ω_X is ample, then $X(F)$ should not be dense in X, but that this could be expected in the opposite case where ω_X^{-1} is ample.

However, as the case of conics already indicates, geometric invariant like the ampleness of ω_X^{-1} cannot suffice to decide on the existence of rational points. For example, the conic C over $\mathbf{Q}$ given by the equation $x^2+y^2+z^2=0$ in the projective plane $\mathbf{P}^2$ has no rational points (a sum of three squares of rational integers cannot be zero, unless they are all zero). However, as soon as the ground field F possesses a square root of -1, then C admits a rational point $P=[1:\sqrt{-1}:0]$ and the usual process of intersecting with the conic C a variable line through that point P gives us a parametrization of $C(F)$.

Smooth projective varieties with ω_X^{-1} are called *Fano varieties.* In general, for such a variety over a number field F, it is expected that *there exists a finite extension F' of F such that $X(F')$ is dense in X for the Zariski topology.* This seems to be a very difficult question to settle in general, unless X has particular properties, like being rational or unirational, in which case a dense set of points in $X(F')$ can be parametrized by the points of a projective space $\mathbf{P}^n(F')$ (where $n=\dim X$).

3.2.3. Eliminating subvarieties. — This conjecture considers the "density aspect" of the rational points. Concerning the counting function, the situation is even more complicated since nothing forbids that most of the rational points of small height are contained in subvarieties.

Let Y be a subscheme of X. One says that Y is *strongly accumulating* if the fraction $N_Y(L,B)/N_X(L,B)$ tends to 1 when $B\to\infty$, and one says that Y is *weakly accumulating* if the inferior limit of this fraction is positive.

As a consequence, the behaviour of the counting function can only be expected to reflect the global geometry if one doesn't count points in accumulating subvarieties. This leads to the notation $N_U(L;B)$ and $Z_U(L;s)$, where U is any Zariski open subset in X.

To get examples of such behaviour, it suffices to blow-up a variety at one (smooth) rational point P. It replaces the point P by a projective space E of dimension $n-1$ if X has dimension n which is likely to possess $\approx B^a$ points of height $\leqslant B$, while the rest could be smaller. For example, if X is an Abelian variety (generalization of elliptic curves in higher dimension), $N_X(L;B)$ only grows like a power of $\log B$.

3.2.4. Definition of a geometric invariant. — In order to predict on a geometric basis the abscissa of convergence of $Z_U(L;s)$, one needs to introduce a function on $\operatorname{Pic}(X)_{\mathbf{R}}$ which is homogeneous of degree 1 and continuous. It is important to know at that point that for Fano varieties, this vector space $\mathrm{Pic}(X)_{\mathbf{R}}$ is finite-dimensional (and identifies with the so-called Néron–Severi group). Indeed, since ω_X^{-1} is ample (this is the very definition of a Fano variety), SERRE's duality and KODAIRA's vanishing theorem imply that $\mathrm{H}^1(X, \mathscr{O}_X) = 0$, so that the Albanese variety of X is trivial.

After considering many examples (Schanuel (1979), Serre (1997), Franke *et al* (1989),...), and with detailed investigations of surfaces at hand, Batyrev & Manin (1990) concluded that the relevant part of $\mathrm{Pic}(X)_{\mathbf{R}}$ was not the ample cone (as the case of the projective space naïvely suggests), but the *effective cone* Λ_{eff}.

In effect, they define for any effective line bundle L a real number $\alpha_X(L)$ which is the least real number a such that $\omega_X \otimes L^{\otimes a}$ belongs to the effective cone in $\mathrm{Pic}(X)_{\mathbf{R}}$.

3.2.5. The conjectures of Batyrev and Manin. — The consideration of the effective cone fits nicely with the elimination of accumulating subvarieties. Indeed, let us assume that the line bundle L belongs to the *interior* of the effective cone; then there exists a positive integer d, an ample line bundle M and an effective line bundle E on X such that $L^{\otimes d} \simeq M \otimes E$. (These line bundles are also called *big.*) For any nonzero section s_E of E, $H_M(x)$ is bounded from below where s_E does not vanish, so that $H_L(x) \gg H_M(x)^{1/d}$ on the open set $U = X \setminus \{s_E = 0\}$. It follows that $N_U(L; B)$ grows at most polynomially, and that the abscissa of convergence $\beta_U(L)$ of $Z_U(L; s)$ is finite.

Batyrev & Manin (1990) present three conjectures of increasing precision concerning the behaviour of the counting function. Let X be a projective smooth variety over a number field F, let L be a line bundle on X which belongs to the interior of the effective cone. The following assertions are conjectured:

(1) For any $\varepsilon > 0$, there exists a dense open subset $U \subset X$ such that $\beta_U(L) \leqslant \alpha_X(L) + \varepsilon$ (*linear growth conjecture*).

(2) If X is a Fano variety, then for any large enough finite extension F' of F, and any small enough dense open set $U \subset X$, one has $\beta_{U,F'}(L) = \alpha(L)$.

(3) Same assertion, only assuming that ω_X does not belong to the effective cone.

3.3. The refined asymptotic expansion

3.3.1. Logarithmic powers. — In the cases evoked above, the height zeta function appears to possess a meromorphic continuation to the left of the line $\mathrm{Re}(s) = \alpha_X(L)$, with a pole of some order t at $s = \alpha_X(L)$. By Tauberian theory, this implies a more precise asymptotic expansion for the counting function, namely $N_U(L; B) \approx B^{\alpha_X(L)}(\log B)^{t-1}$. A stronger form of the conjecture of BATYREV–MANIN also predicts the order of this pole, at least when $L = \omega_X^{-1}$. In that case, BATYREV and MANIN conjecture that t is the dimension of the real vector space $\mathrm{Pic}(X)_{\mathbf{R}}$. (Recall that the Picard group of a Fano variety has finite rank.)

3.3.2. (In)compatibilities. — Some basic checks concerning that conjecture had been made by Franke *et al* (1989), for instance its compatibility with products of varieties. However, Batyrev & Tschinkel (1996*b*) observed that *the conjecture is not compatible with families* and produced a *counterexample.*

They consider the subscheme V of $\mathbf{P}^3 \times \mathbf{P}^3$ defined by the equation $x_0 y_0^3 + \cdots + x_3 y_3^3 = 0$ (where $[x_0 : \ldots : x_3]$ are the homogeneous coordinates on the first factor, and $[y_0 : \ldots : y_3]$ are those on the second). For fixed $\mathbf{x} \in \mathbf{P}^3$, the fibre $V_{\mathbf{x}}$ consisting of $\mathbf{y} \in \mathbf{P}^3$ such that $(\mathbf{x}, \mathbf{y}) \in V$ is a diagonal cubic surface; in other words, V is the total space of the family of diagonal cubic surfaces.

Let p_1 and p_2 be the two projections from $\mathbf{P}^3 \times \mathbf{P}^3$ to $\mathbf{P}^3$. For any couple of integers (a, b), let $\mathscr{O}(a, b)$ be the line bundle $p_1^* \mathscr{O}_{\mathbf{P}^3}(a) \otimes p_2^* \mathscr{O}_{\mathbf{P}^3}(b)$ on $\mathbf{P}^3 \times \mathbf{P}^3$. The so-defined map from $\mathbf{Z}^2$ to $\operatorname{Pic}(\mathbf{P}^3 \times \mathbf{P}^3)$ is an isomorphism of abelian groups. Moreover, since V is an hypersurface in $\mathbf{P}^3 \times \mathbf{P}^3$, the Lefschetz theorem implies that the restriction map induces an isomorphism from $\operatorname{Pic}(\mathbf{P}^3 \times \mathbf{P}^3)$ to $\operatorname{Pic}(V)$. The anticanical line bundle ω_V^{-1} of V corresponds to $\mathscr{O}(3, 1)$ (anticanonical of $\mathbf{P}^3 \times \mathbf{P}^3$ minus the class of the equation, that is $(4, 4) - (1, 3)$) and the conjecture of Batyrev–Manin predicts that there should be $\approx B(\log B)$ points of $\mathscr{O}(3, 1)$-height $\leqslant B$ in $V(F)$.

Let us fix some $\mathbf{x} \in \mathbf{P}^3$ such that $V_{\mathbf{x}}$ is non-singular. This is a cubic surface, and its anticanonical line bundle identifies with the restriction of $\mathscr{O}_{\mathbf{P}^3}(1)$. Thus, if $t_{\mathbf{x}} = \dim_{\mathbf{R}} \operatorname{Pic}(V_{\mathbf{x}})_{\mathbf{R}}$, the conjecture predicts that there should be $\approx B(\log B)^{t_{\mathbf{x}}}$ points of $\mathscr{O}(1)$-height $\leqslant B$ in $V_{\mathbf{x}}(F)$.

As a consequence, if the conjecture holds for V and if $t_{\mathbf{x}} > 2$, then the fibre $V_{\mathbf{x}}$ would have more points than what the total space seem to have. This doesn't quite contradict the conjecture however, because it explicitly allows to remove a finite union of subvarieties.

However, the rank of the Picard group can exhibit disordered behaviour in families; for example, it may not be semi-continuous, and jump on a infinite union of subvarieties. This happens here, since $t_{\mathbf{x}} = 7$ when F contains the cubic roots of unity and all the homogeneous coordinates of $\mathbf{x}$ are cubes. The truth of the conjecture for V therefore requires to omit all such fibres $V_{\mathbf{x}}$. But they form an infinite union of disjoint subvarieties, a kind of accumulating subset which is not predicted by the conjecture of Batyrev–Manin.

In particular, this conjecture is false, either for V, or for most cubic surfaces. By geometric considerations, and using their previous work on toric varieties, Batyrev & Tschinkel (1996*b*) could in fact conclude that the conjecture does not hold for V.

3.3.3. Peyre's refinement of the conjecture. — One owes to Peyre (1995) a precise refinement of the previous conjecture, as well as the verification of the refined conjecture in many important cases. Indeed, all known positive examples feature an asymptotic expansion of the form $N_U(\omega_X^{-1}; B) \sim cB(\log B)^t$, for some positive real number c, which in turn is the product of four factors:

(1) the volume of a suitable subspace $X(\mathbb{A}_F)^{\mathrm{B}}$ of the adelic space $X(\mathbb{A}_F)$ (where the so-called Brauer-Manin obstruction to rational points vanishes) with respect to the *Tamagawa measure* first introduced in that context by Peyre (1995).

(2) the cardinality of the finite Galois cohomology group $\mathrm{H}^1(\operatorname{Pic}(X_{\overline{F}}))$;

(3) a rational number related to the location of the anticanonical line bundle in the effective cone Λ_{eff} of $\operatorname{Pic}(X)_{\mathbf{R}}$;
(4) the rational number $1/t!$.

Let us give a short account of Peyre's construction of a measure. The idea is to use the given adelic metric on ω_X^{-1} to construct a measure $\tau_{X,v}$ on the local spaces $X(F_v)$ for all places v of the number field F, and then to consider a suitable (renormalized) product of these measures.

3.3.4. Local measures. — Ideally, a measure on an n-dimensional analytic manifold is given in local coordinates by an expression of the form $\mathrm{d}\mu(x) = f(x)\,\mathrm{d}x_1 \dots \mathrm{d}x_n$, where f is a positive function. When one considers another system of local coordinates, the expression of the measure changes, and its modification is dictated by the change-of-variables formula for multiple integrals: the appearance of the absolute value of the Jacobian means that $\mathrm{d}\mu(x)$ is modified as if it were "the absolute value of a differential form". One can deduce from this observation that if α is a local (*i.e.*, in a chart) differential n-form, written $\alpha = f(x)\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$ in coordinates, then the measure

$$\frac{|\alpha|_v}{\|\alpha\|_v} = \frac{|f(x)|_v}{\|\alpha\|_v}\,\mathrm{d}x_1 \dots \mathrm{d}x_n$$

is well-defined, independently of the choice of α. One can therefore glue all these local measures and obtain a measure $\tau_{X,v}$ on $X(F_v)$.

3.3.5. Convergence of an infinite product. — Since X is proper, the adelic space $X(\mathbb{A}_{\mathbf{Q}})$ is the (infinite) product of all $X(F_v)$, for $v \in \operatorname{Val}(F)$, endowed with the product topology. To define a measure on $X(\mathbb{A}_{\mathbf{Q}})$ one would like to consider the (infinite) product of the measures $\tau_{X,v}$. However, this gives a finite measure if and only if the product

$$\prod_{v \text{ finite}} \tau_{X,v}(X(F_v))$$

converges absolutely. This convergence never holds, and one therefore needs to introduce *convergence factors* to define a measure on $X(\mathbb{A}_{\mathbf{Q}})$.

By a formula of Weil (1982), the equality $\tau_{X,v}(X(F_v)) = q_v^{-\dim X} \operatorname{Card}(X(k_v))$ holds for almost all finite places v, where k_v is the residue field of F at v, q_v is its cardinality, and $X(k_v)$ is the set of solutions in k_v of a fixed system of equations with coefficients in the ring of integers of F which defines X. By Weil's conjecture, established by Deligne (1974), plus various cohomological computations, this implies that

$$\tau_{X,v}(X(F_v)) = 1 + \frac{1}{q_v} \operatorname{Tr}(\operatorname{Frob}_v \mid \operatorname{Pic}(X_{\overline{F}})_{\mathbf{R}}) + \mathrm{O}(q_v^{-3/2}),$$

where Frob_v is a "geometric Frobenius element" of the Galois group of $\overline{F}$ at the place v.

Let us consider the Artin L-function of the Galois-module $P = \mathrm{Pic}(X_{\overline{F}})_{\mathbf{R}}$: it is defined as the infinite product

$$\mathrm{L}(s,P) = \prod_{v \text{ finite}} \mathrm{L}_v(s,P), \qquad \mathrm{L}_v(s,P) = \det(1 - q_v^{-s}\,\mathrm{Frob}_v\,|P)^{-1}.$$

The product converges absolutely for $\mathrm{Re}(s) > 1$, defines a holomorphic function in that domain. Moreover, $\mathrm{L}(s,P)$ has a meromorphic continuation to $\mathbf{C}$ with a pole of order $t = \dim \mathrm{Pic}(X)_{\mathbf{R}}$ at $s = 1$. Let

$$\mathrm{L}^*(1,P) = \lim_{s\to 1} \mathrm{L}(s,P)(s-1)^{-t};$$

it is a positive real number.

Comparing the asymptotic expansions for $\mathrm{L}_v(1,P)$ and $\tau_{X,v}(X(F_v))$, one can conclude that the following infinite product

$$\tau_X = \mathrm{L}^*(1,P) \prod_v \big(\mathrm{L}_v(1,P)^{-1}\tau_{X,v}\big)$$

is absolutely convergent and defines a Radon measure on the space $X(\mathbb{A}_F)$.

3.3.6. Equidistribution. — One important aspect of the language of metrized line bundles is that it suggests explicitly to look at what happens when one changes the adelic metric on the canonical line bundle.

Assume for example that there is a number field F and an open subset U such that Manin's conjecture (with precised constant as above) holds for *any* adelic metric on ω_X. Then, Peyre (1995) shows that rational points in $U(F)$ of heights $\leqslant B$ *equidistribute* towards the probability measure on $X(\mathbb{A}_F)^{\mathrm{B}}$ proportional to τ_X. Namely, for any smooth function Φ on $X(\mathbb{A}_F)$,

$$\frac{1}{N_U(\omega_X^{-1};B)} \sum_{\substack{x\in U(F)\\ H_{\omega_X^{-1}}(x)\leqslant B}} \Phi(x) \longrightarrow \frac{1}{\tau_X(X(\mathbb{A}_F)^{\mathrm{B}})} \int_{X(\mathbb{A}_F)^{\mathrm{B}}} \Phi(x)\,\mathrm{d}\tau_X(x).$$

As we shall see in the next section, this strengthening by Peyre of the conjecture of Batyrev–Manin is true in many remarkable cases, with quite nontrivial proofs.

4. Methods and results

4.1. Explicit counting

4.1.1. Projective space. — Let F be a number field and let L be the line bundle $\mathscr{O}(1)$ on the projective space $\mathbf{P}^n$, with an adelic metric. Schanuel (1979) has given the following asymptotic expansion for the counting function on $\mathbf{P}^n$:

$$N_{\mathbf{P}^n}(L;B) \sim \frac{\mathrm{Res}_{s=1}\,\zeta_F(s)}{\zeta_F(n+1)} \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|D_F|}}\right)^n (n+1)^{r_1+r_2-1} B^{n+1}.$$

In this formula, ζ_F is the Dedekind zeta function of the field F, r_1 and $2r_2$ are the number of real embeddings and complex embeddings of F, and D_F is its discriminant.

In fact, it has been shown later by many authors, see *e.g.* Franke *et al* (1989), that the height zeta function $Z_{\mathbf{P}^n}(L; s)$ is holomorphic for $\mathrm{Re}(s) > n + 1$, has a meromorphic continuation to the whole complex plane $\mathbf{C}$, with a pole of order 1 at $s = n + 1$ and no other pole on the line $\{\mathrm{Re}(s) = 1\}$.

To prove this estimate, one can sort points $[x_0 : \ldots : x_n]$ in $\mathbf{P}^n(F)$ according to the class of the fractional ideal generated by $(x_0, \ldots, x_n)$. Constructing fundamental domains for the action of units, the enumeration of such sets can be reduced to the counting of lattice points in homothetic sets in $\mathbf{R}^{(n+1)(r_1+2r_2)}$ whose frontier has smaller dimension. The Möbius inversion formula is finally used to take care of the coprimality condition.

4.1.2. Hirzebruch surfaces and other blow-ups. — There are many other cases where this explicit method works. Let us mention only some of them:

– ruled surfaces over the projective line (Hirzebruch surfaces);

– Grassmann varieties (Thunder (1992)) ;

– Del Pezzo surfaces, given as blow-ups of the plane in a few points in general position (de la Bretèche (2002));

– some Chow varieties of $\mathbf{P}^n$, see Schmidt (1993).

This method has two major disadvantages: 1) it is hard to take advantage of subtle geometric properties of the situation studied; 2) it requires to *know* already much about the rational points, *e.g.*, to be able to parametrize them. However, this is so far the only way of dealing with the Del Pezzo surfaces which are not toric. It often requires important tools of algebraic geometry, like the description of universal torsors, or the more-or-less equivalent computation of homogeneous coordinate rings, see Colliot-Thélène & Sansuc (1987) and Cox (1995).

4.2. The circle method of Hardy–Littlewood

The *circle method* was initially devised to tackle Waring's problem, namely the decomposition of an integer into sums of powers. More generally, it is well suited to the study of diophantine equations in "many variables".

Concerning our counting problem, it indeed allows to establish the conjecture of Batyrev–Manin–Peyre for smooth complete intersections of codimension m in $\mathbf{P}^n$, that is, subschemes X of $\mathbf{P}^n$ defined by the vanishing of m homogeneous polynomials $f_1, \ldots, f_m$ such that at every point $x \in X$, the Jacobian matrix of the f_i has rank m. Let $d_1, \ldots, d_m$ denote the degrees of the polynomials $f_1, \ldots, f_m$; then the canonical line bundle of X is the restriction to X of the line bundle $\mathscr{O}(-n-1+\sum_{i=1}^m d_i)$. Consequently, X is Fano if and only if $d_1 + \cdots + d_m \leqslant n$. Therefore, specific examples of Fano complete intersections are lines or conics in $\mathbf{P}^2$, planes, quadrics or cubic surfaces in $\mathbf{P}^3$, etc.

The circle method is restricted to cases where the number of variables n is very large in comparison to the degrees $d_1, \ldots, d_m$. For instance, Birch (1962) establishes the desired result for an hypersurface of degree d in $\mathbf{P}^n$ provided $n \geqslant 2^d(d-1)$. However, when it applies, it gives very strong results, including a form of the Hasse principle and equidistribution, see Peyre (1995).

4.3. The Fourier method

The remaining methods that we describe now are very powerful but limited to classes of varieties possessing a "large" action of an algebraic group.

We begin by the Fourier method, by which we mean the generalization of the theory of Fourier series and Fourier integrals (which concern functions on the groups $\mathbf{R}/\mathbf{Z}$ and $\mathbf{R}$) to more general topological groups.

4.3.1. Algebraic groups and their compactifications. — Let X be an *equivariant compactification* of an algebraic group G over a number field F. This means that X is a projective variety containing G as a dense open subset and that the left and right actions of G on itself extend to actions on X.

Then, $G(F)$ is a discrete subgroup of the adelic group $G(\mathbb{A}_F)$. Moreover, for line bundles L on X, the height H_L can often be extended to a function on $G(\mathbb{A}_F)$: there are functions $H_{L,v}$ such that, for $\mathbf{g} \in G(F) \subset G(\mathbb{A}_F)$,

$$H_L(\mathbf{g}) = \prod_{v \in \mathrm{Val}(F)} H_{L,v}(g_v).$$

Then, the height zeta function $Z_G(L; s)$ is an average over the discrete group $G(F)$ of a function H_L^{-s} on $G(\mathbb{A}_F)$ and one can use *harmonic analysis* and the spectral decomposition to understand $Z_G(L; s)$.

Important cases of groups in which mathematicians have been able to use harmonic analysis to prove the conjectures of Batyrev–Manin–Peyre are the following.

– *algebraic tori*: then X is called a *toric variety*, see Batyrev & Tschinkel (1996*a*);
– *vector groups*, Chambert-Loir & Tschinkel (2002);
– the Heisenberg group of upper triangular matrices, Shalika & Tschinkel (2004);
– reductive groups, embedded in the *wonderful compactification* defined by De Concini–Procesi, see Shalika *et al* (2007).

4.3.2. The Poisson formula. — The simplest case is when G is commutative because irreducible representations are one-dimensional (*i.e.*, are *characters*). Let us fix some notation. Let μ be a Haar measure on the locally compact group $G(\mathbb{A}_F)$, given as product of suitable Haar measures μ_v on $G(F_v)$. Let $\widehat{G_v} = \mathrm{Hom}(G(F_v), \mathbf{U})$ be the group of characters of $G(F_v)$ and $\widehat{G(\mathbb{A}_F)} = \mathrm{Hom}(G(\mathbb{A}_F), \mathbf{U})$ be the group of characters of $G(\mathbb{A}_F)$, where $\mathbf{U}$ is the group of complex numbers of modulus 1. By restriction, any character $\chi \in \widehat{G(\mathbb{A}_F)}$ induces a character χ_v of $G(F_v)$, for each place v. Let $G(F)^\perp$ be the orthogonal of $G(F)$, that is the group of characters of $G(\mathbb{A}_F)$ which are trivial on $G(F)$. This is a locally compact group and it carries a dual measure $\mu^\perp$.

Any integrable function $f \in L^1(G(F_v))$ has a *Fourier transform*, which is the function on $\widehat{G(F_v)}$ defined as follows: for $\chi \in \widehat{G(\mathbb{A}_F)}$,

$$\mathscr{F}(f;\chi) = \int_{G(\mathbb{A}_F)} f(\mathbf{g})\chi(\mathbf{g})\,\mathrm{d}\mu(\mathbf{g}).$$

If f is a simple function, *i.e.*, of the form $f(\mathbf{g}) = \prod f_v(g_v)$, then $\mathscr{F}(f;\chi)$ is a product $\prod \mathscr{F}_v(f_v;\chi_v)$ of local Fourier transforms defined in an analogous manner.

The generalization of *Poisson formula* to this context states that if f is an integrable function on $G(\mathbb{A}_F)$ such that moreover $\mathscr{F}(f;\cdot)$ is integrable on $G(F)^\perp$, then

$$\sum_{g\in G(F)} f(g) = \int_{G(F)^\perp} \mathscr{F}(f;\chi)\,\mathrm{d}\mu^\perp(\chi).$$

This gives a formula for the height zeta function

$$Z_G(L;s) = \int_{G(F)^\perp} \mathscr{F}(H_L^{-s};\chi)\,\mathrm{d}\mu^\perp(\chi),$$

which in many cases proved to be a key tool towards establishing the desired analytic properties of the height zeta function.

4.3.3. The Fourier transform at the trivial character. — Let us consider the Fourier transform of $f = H_L^{-s}$ at the trivial character, in other words, the function

$$\Phi(s) = \mathscr{F}(H_L^{-s},\mathbf{1}) = \int_{G(\mathbb{A}_F)} H_L(g)^{-s}\,\mathrm{d}\mu(g).$$

The decomposition of the function H_L as a product of function on the spaces $G(F_v)$ implies a similar decomposition of Φ as a the product $\Phi(s) = \prod_v \Phi_v(s)$, where

$$\Phi_v(s) = \mathscr{F}_v(H_L^{-s},\mathbf{1}) = \int_{G(F_v)} H_L(g)^{-s}\,d\mu_v(g).$$

In the above mentioned cases, these functions have been often computed explicitly, resorting to properties of algebraic groups. However, this can also be done in a very geometric manner thanks to the *fundamental observation*: the integral $\Phi_v(s)$ on $G(F_v)$ can be viewed as a geometric analogue on $X(F_v)$ of Igusa's local zeta functions. Assume for simplicity that $L = \omega_X^{-1}$. In Section 5 we will explain how to prove the following facts (see § 5.2.6):

– $\Phi_v(s)$ converges for $\operatorname{Re}(s) > 0$;
– $\Phi_v(s)$ has a meromorphic continuation to $\mathbf{C}$;
– for almost all places v, $\Phi_v(s)$ can be explicitly computed (Denef's formula);
– the product $\prod \Phi_v(s)$ converges for $\operatorname{Re}(s) > 1$;
– this product has a meromorphic continuation which is governed by some Artin L-function.

We shall in fact explain that these facts apply in much more generality than that of algebraic groups.

4.3.4. Conclusion of the proof. — To prove MANIN's conjecture for an equivariant compactification of an algebraic group using the Fourier method, a lot of work still remains to be done. One first has to establish a similar meromorphic continuation for other characters (other representations in the non-commutative cases) and, then, to integrate all these contributions over $G(F)^{\perp}$. Of course, the last step requires that one obtains good upper bounds in the first step, so that integration is at all possible. Details depend however very much on the specific groups involved and cannot be described here.

4.4. More harmonic analysis

4.4.1. Ergodic analysis. — These methods take into account the *dynamical* properties of an action of a discrete or Lie group in the situation studied. Let us give two examples:

– algebraic groups themselves, like SL_n viewed as the subset of $\mathbf{P}^{n^2}$ defined by the equation $z^n = \det(A)$, where a point of $\mathbf{P}^{n^2}$ is a non-zero pair $(z, A) \in F \times \mathrm{Mat}_n(F)$ modulo homotheties.

– homogeneous spaces of algebraic groups, *e.g.*, the set V_P of matrices with fixed characteristic polynomial. When P has distinct roots, any two such matrices are conjugate, hence V_P is an homogeneous space of GL_n.

Methods of ergodic theory apply usually when one considers *lattices* of real or adelic semi-simple groups (that is, discrete subgroups of cofinite volume). Keywords are the equidistribution theorem of M. RATNER for unipotent flows (used in the closely related context of integral points by Eskin & McMullen (1993); Eskin *et al* (1996)) or, as in Gorodnik *et al* (2009), mixing properties of the dynamical system, implied by decay of matrix coefficients of unitary representations in the spirit of HOWE–MOORE.

These methods allow to derive asymptotic expansions for the counting function by comparing it to the volume of a corresponding set in a real or adelic space, whose asymptotic growth has to be established independently. They are also amenable to proving equidistribution results. However, up to now, they do not establish the analytic properties of the height zeta function.

4.4.2. Eisenstein series. — *Flag varieties* are classical generalizations of the projective space: they parametrizes subvector spaces W of a fixed vector space V (Grassmann variety), of, more generally, increasing families $W_1 \subset \cdots \subset W_m$ ("flags") of subvector spaces. Let us consider for simplicity the case of a fixed vector space V of dimension n. and the Grassmann variety Gr_d^n of subspaces of fixed dimension d. The projective space $\mathbf{P}(V)$ is recovered by considering the cases $d = 1$ (parametrizing lines) and $d = n - 1$ (parametrizing hyperplanes). Let us fix a specific subspace W_0 of dimension d in V; considering a basis of W_0 and extending it to a basis of V, it is easy to observe that any other subspace of dimension d is of the form gW_0, for some automorphism $g \in \mathrm{GL}(V)$; moreover, $gW_0 = g'W_0$ if and only if $g^{-1}g'W_0 = W_0$, that is, if $g^{-1}g'$ belongs to the *stabilizer* P of W_0 for the action of $\mathrm{GL}(V)$. In the basis that we fixed, P is the set of invertible upper-triangular block matrices having only zeroes in the lower rectangle $[d+1,\ldots,n]\times[1,\ldots,d]$.

More generally, in the language of algebraic groups, generalized flag varieties are quotients of reductive groups by parabolic subgroups. The understanding of their height zeta functions is a consequence of the observation, due to J. Franke, that these functions were studied extensively in the theory of automorphic forms, being nothing else than generalized *Eisenstein series.* Modulo some (not so obvious) translation, the results of R. P. Langlands on these Eisenstein series readily imply the conjectures of Batyrev–Manin and Peyre (including equidistribution) for the flag varieties, see Franke *et al* (1989) and Peyre (1995).

5. Heights and Igusa zeta functions

This final section aims at explaining the *geometric computation* of the Fourier transform of the height function via the theory of Igusa zeta functions. It essentially borrows from the introduction of our recent article Chambert-Loir & Tschinkel (2008).

5.1. Heights and measures on adelic spaces

5.1.1. Heights. — Let X be a projective variety over a number field F. Let L be an effective divisor on X, and let us endow the associated line bundle $\mathscr{O}_X(L)$ with an adelic metric. This line bundle possesses a canonical global section f_L whose divisor is L. For any place $v\in\operatorname{Val}(F)$, the function $x\mapsto \|\mathsf{f}_L(x)\|_v$ is continuous on $X(F_v)$ and vanishes precisely on L. Moreover, its sup-norm is equal to 1 for almost all places v.

Consequently, the infinite product $\prod_{v\in\operatorname{Val}(F)}\|\mathsf{f}_L(x_v)\|_v$ converges to an element of $\mathbf{R}_+$ for any adelic point $x=(x_v)\in X(\mathbb{A}_F)$. Let $U=X\setminus L$. If x is an adelic point of U, then $\|\mathsf{f}_L(x_v)\|_v\neq 0$ for all v, and is in fact equal to 1 for almost all v. We thus can define the *height* of a point $x\in U(\mathbb{A}_F)$ to be equal to

$$H_L((x_v))=\prod_{v\in\operatorname{Val}(F)}\|\mathsf{f}_L(x_v)\|_v^{-1}.$$

The resulting function on $U(\mathbb{A}_F)$ is continuous and admits a positive lower bound. Moreover, for any $B>0$, the set of points $x\in U(\mathbb{A}_F)$ such that $H_L(x)\leqslant B$ is compact.

5.1.2. Local measures (II). — The measure $\tau_{X,v}$ on $X(F_v)$ defined in § 3.3.4 gives finite volume to $X(F_v)$ and the open set $U(F_v)$ has full measure. We modify the construction as follows so as to define a Radon measure on $U(F_v)$ whose total mass (unless $L(F_v)=\emptyset$) is now infinite:

$$\mathrm{d}\tau_{(X,L),v}(x)=\frac{1}{\|\mathsf{f}_L(x)\|_v}\mathrm{d}\tau_{X,v}(x).$$

For example, in the important case where X is an equivariant compactification of an algebraic group G, $U = G$ and $-L$ is a canonical divisor, this measure is a Haar measure on the locally compact group $G(F_v)$.

5.1.3. Definition of a global Tamagawa measure. — The product of these local measures does not converge in general, and to use them to construct a measure on the adelic space $U(\mathbb{A}_F)$, it is necessary to introduce convergence factors, *i.e.*, a family (λ_v) of positive real numbers such that the product

$$\prod_{v \text{ finite}} \lambda_v \tau_{(X,L),v}(U(\mathfrak{o}_v))$$

converges absolutely. The limit will thus be a positive real number (unless some factor $U(\mathfrak{o}_v)$ is empty, which does not happen if we remove an adequate finite set of places in the product). The existence of such factors is a mere triviality, since one could take for λ_v the inverse of $\tau_{(X,L),v}(U(\mathfrak{o}_v))$. Of course, we claim for a meaningful definition of a family (λ_v), based on geometric or arithmetic invariants of U.

The condition under which we can produce such factors is the following: *X is proper, smooth, geometrically connected, and the two cohomology groups $\mathrm{H}^1(X, \mathscr{O}_X)$ and $\mathrm{H}^2(X, \mathscr{O}_X)$ vanish.* Let us assume that this holds and let us define

$$M^0 = \mathrm{H}^0(U_{\overline{F}}, \mathscr{O}^*)/\overline{F}^* \quad \text{and} \quad M^1 = \mathrm{H}^1(U_{\overline{F}}, \mathscr{O}^*)/\text{torsion};$$

in words, M^0 is the abelian groups of invertible functions on $U_{\overline{F}}$ modulo constants, and M^1 is the Picard group of $U_{\overline{F}}$ modulo torsion. By the very definition as a cohomology group of $U_{\overline{F}}$, they possess a canonical action of the Galois group Γ_F of $\overline{F}/F$. Moreover, M^0 is a free $\mathbf{Z}$-module of finite rank. Indeed, to an invertible function on $U_{\overline{F}}$, one may attach its divisor, which is supported by the complementary subset, that is $L_{\overline{F}}$. This gives a morphism from M^0 to the free abelian group generated by the irreducible components of $L_{\overline{F}}$ and this map is injective because X is normal. It follows that M^0 is a free $\mathbf{Z}$-module of finite rank. Moreover, under the vanishing assertion above, then M^1 is a free $\mathbf{Z}$-module of finite rank too. We thus can consider the Artin L-functions of these two Γ_F-modules, $\mathrm{L}(s, M^0)$ and $\mathrm{L}(s, M^1)$.

Using Weil's conjectures, proved by Deligne (1974), to estimate the volumes $U(\mathfrak{o}_v)$, in a similar manner to what Peyre had done to define the global measure τ_X on $X(\mathbb{A}_F)$, we prove that the family (λ_v) given by

$$\lambda_v = \frac{\mathrm{L}_v(1, M^0)}{\mathrm{L}_v(1, M^1)}$$

if v is a finite place of F and $\lambda_v = 1$ if v is archimedean is a family of convergence factors. In other words, the infinite product of measures

$$\frac{\mathrm{L}^*(1, M^1)}{\mathrm{L}^*(1, M^0)} \prod_{v \in \mathrm{Val}(F)} \frac{\mathrm{L}_v(1, M^0)}{\mathrm{L}_v(1, M^1)} \, \mathrm{d}\tau_{(X,L),v}(x_v)$$

defines a Radon measure on the locally compact space $U(\mathbb{A}_F)$. We denote this measure by $\tau_{(X,L)}$ and call it the *Tamagawa measure* on $U(\mathbb{A}_F)$.

5.1.4. Examples for the Tamagawa measure. — This construction coincides with PEYRE's construction if $L = \emptyset$, since then $M^0 = 0$ (there are no non-constant invertible functions on X) and $M^1 = \mathrm{Pic}(X_{\overline{F}})/\text{torsion}$.

It also coincides with the classical constructions for algebraic groups. Let us indeed assume that X is an equivariant compactification of an algebraic group G and that $-L$ is a canonical divisor, with $G = U = X \setminus L$. We have mentionned that $\tau_{(X,L),v}$ is a Haar measure on $G(F_v)$. If G is semi-simple, or if G is nilpotent, then $M^0 = M^1 = 0$; in that case $\lambda_v = 1$, according to the fact that the global measure does not need any convergence factor. However, if G is a torus, then M^0 is the group of characters of $G_{\overline{F}}$ and $M^1 = 0$; our definition coincides with ONO's. In all of these cases, the global measure we construct is of course a Haar measure on the locally compact group $G(\mathbb{A}_F)$.

It also recovers some cases of homogeneous spaces G/H, where G and H are semisimple groups over $\mathbf{Q}$, under the assumption that H has no nontrivial characters, like those studied by Borovoi & Rudnick (1995). In fact, the definition of Tamagawa measures on such homogeneous spaces does not need any convergence factors, as can be seen by the fact that $M^0 = 0$ (since invertible functions on G are already constants) and $M^1 = 0$ (because the Picard group of G/H is given by characters of H).

As a last example, let us recall that Salberger (1998) had shown that the universal torsors over toric varieties do not need any convergence factors, because the convergence factor of the fibers (a torus) compensates the one of the base (a projective toric variety). In our approach, this is accounted by the fact, discovered by Colliot-Thélène & Sansuc (1987), that such torsors have neither non-constant invertible functions, nor non-trivial line bundles, hence $M^0 = M^1 = 0$ and $\lambda_v = 1$.

5.2. An adelic-geometric analogue of Igusa's local zeta functions

As we explained above, when using the Fourier method to elucidate the number of points of bounded height on some algebraic varieties, we are lead to establish the meromorphic continuation of the function of a complex variable:

$$Z\colon s \mapsto \int_{U(\mathbf{A}_F)} H_L(x)^{-s}\,\mathrm{d}\tau_{(X,L)}(x).$$

(This function was denoted Φ in §4.3.) By definition, the measure $\tau_{(X,L)}$ is (up to the convergence factors and the global factor coming from Artin L-functions) the product of the local measures $\tau_{(X,L),v}$ on the analytic manifolds $U(F_v)$. Similarly, the integrated function $x \mapsto H_L(x)^{-s}$ is the product of the local functions $x \mapsto \|\mathsf{f}_L(x)\|_v^s$. Consequently, provided it converges absolutely, this adelic integral decomposes as a product of integrals over local fields,

$$Z(s) = \frac{\mathrm{L}^*(1,M^1)}{\mathrm{L}^*(1,M^0)} \prod_{v\in\mathrm{Val}(F)} \frac{\mathrm{L}_v(1,M^0)}{\mathrm{L}_v(1,M^1)} Z_v(s), \qquad Z_v(s) = \int_{U(F_v)} \|\mathsf{f}_L(x)\|_v^s \,\mathrm{d}\tau_{(X,L),v}(x).$$

5.2.1. Geometric Igusa zeta functions. — IGUSA's theory of local zeta functions studies the analytic properties of integrals of the form

$$I_\Phi(s) = \int_{F_v^n} \Phi(x)\, |f(x)|_v^s \,\mathrm{d}x,$$

where f is a polynomial and Φ is a smooth function with compact support on the affine n-space over F_v. (See Igusa (2000) for a very good survey on this theory.)

Our integrals $Z_v(s)$ are straightforward generalizations of IGUSA's local zeta functions. Indeed, since $\tau_{(X,L),v} = \|\mathsf{f}_L\|_v^{-1}\tau_{X,v}$, we have

$$Z_v(s) = \int_{U(F_v)} \|\mathsf{f}_L(x)\|_v^{s-1} \,\mathrm{d}\tau_{X,v}(x) = \int_{X(F_v)} \|\mathsf{f}_L(x)\|_v^{s-1} \,\mathrm{d}\tau_{X,v}(x)$$

because $L(F_v)$ has measure 0 in $X(F_v)$. In our case, we integrate on a compact analytic manifold, rather than a test function on the affine space; as always in Differential geometry, partitions of unity and local charts convert a global integral into a finite sum of local ones. The absolute value of a polynomial has been replaced by the norm of a global section of a line bundle; again, in local coordinates, the function $\|\mathsf{f}_L\|_v$ is of the form $|f|_v\,\varphi$, where f is a local equation of L and φ a smooth non-vanishing function. However, because of these slight differences, we call such integrals *geometric Igusa zeta functions*.

From now on, we will assume in this text that over the algebraic closure $\overline{F}$, L is a divisor with simple normal crossings. This means that the irreducible components of the divisor $L_{\overline{F}}$ are smooth and meet transversally; in particular, any intersection of part of these irreducible components is either empty, or a smooth subvariety of the expected codimension. This assumption is only here for the convenience of the computation. Indeed, by the theorem on resolution of singularities of Hironaka (1964), there exists a proper birational morphism $\pi\colon Y \to X$, with Y smooth, which is an isomorphism on U and such that the inverse image of L (as a Cartier divisor) satisfies this assumption. We may replace (X, L) by (Y, π^*L) without altering the definitions of Z, Z_v, etc.. Let $\mathscr{A}$ be the set of irreducible components of L; for $\alpha \in \mathscr{A}$, let L_α be the corresponding component, and d_α its multiplicity in L. We thus have $L = \sum_{\alpha\in\mathscr{A}_v} d_\alpha L_\alpha$.

This induces a stratification of X indexed by subsets $A \subset \mathscr{A}$, the stratum X_A being given by

$$X_A = \{x \in X\,;\, x \in L_\alpha \Leftrightarrow \alpha \in A\}.$$

In the sequel, we shall moreover assume that for any $\alpha \in \mathscr{A}$, L_α is geometrically irreducible. This does not cover all cases, but the general case can be treated using a similar analysis, the Dedekind zeta function of F being replaced by Dedekind zeta functions of finite extensions F_α defined by the components L_α; I refer the interested reader to Chambert-Loir & Tschinkel (2008) for details.

5.2.2. Local analysis. — Let x be a point in $X_A(F_v)$. The "normal crossings" assumption on L implies that on a small enough analytic neighbourhood Ω_x of x, one may find local equations x_α of L_α, for $\alpha \in A$, and complete these local equations into a system of local coordinates $((x_\alpha)_{\alpha\in A}; y_1, \dots, y_s)$, so that $s+\mathrm{Card}(A) = \dim X$. In this neighbourhood of x, the measure τ can be written $\omega(x;y)\,\mathrm{d}x\,\mathrm{d}y$, with ω a positive continuous function, and there is a continous and positive function φ such that

$$\|\mathsf{f}_L\|_v = \varphi\cdot\prod_{\alpha\in A}|x_\alpha|_v^{d_\alpha}$$

on Ω_x. If Φ_x is a smooth function with compact support on Ω_x, we thus have

$$\int_{X(F_v)}\Phi_x\,\|\mathsf{f}_L\|^s\;\tau_{(X,L),v} = \int_{\Omega_x}(\varphi^s\omega\Phi_x)\prod_{\alpha\in A}|x_\alpha|_v^{(s-1)d_\alpha}\prod_{\alpha\in A}\mathrm{d}x_\alpha\,\mathrm{d}y_1\dots\mathrm{d}y_s.$$

By comparison with the integrals

$$\int_{F_v}\Phi(u)\,|u|_v^s\;\mathrm{d}u$$

which converges for $\mathrm{Re}(s) > -1$ if Φ is compactly supported on F_v, we conclude that

$$\int_{X(F_v)}\Phi_x\,\|\mathsf{f}_L\|^s\;\tau_{(X,L),v}$$

converges absolutely for $\mathrm{Re}(s) > 0$.

Introducing a partition of unity, we conclude that for any place $v\in\mathrm{Val}(F)$, the integral $Z_v(s)$ converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function on this half-plane.

5.2.3. Denef's formula. — Our understanding of the infinite product of the functions $Z_v(s)$ relies in the explicit computation of almost all of them. The formula below is a straightforward generalization of a formula which Denef (1987) used to prove that, when f is a polynomial with coefficients in the number field F and Φ is the characteristic function of the unit polydisk in F_v^n, then IGUSA's local zeta function $I_\Phi(s)$ is a rational function of q_v^{-s} whose degree is bounded when v varies within the set of finite places of F.

Assuming that the whole situation has "good reduction" at a place v, we obtain the following formula, where k_v is the residue field of F at v and q_v is its cardinality:

$$Z_v(s) = \sum_{A\subset\mathscr{A}} q_v^{-\dim X}\,\mathrm{Card}(X_A(k_v)).\prod_{\alpha\in A}\frac{q_v-1}{q_v^{1+d_\alpha(s-1)}-1}.$$

As in Denef (1987), this is a consequence of the fact that for each point $\tilde{x}\in X(k_v)$, in the local analytic description of the previous paragraph, we may find local coordinates x_α and y such that $\varphi\equiv\Omega = 1$ which parametrize the open unit polydisk with "center $\tilde{x}$." That this is at all possible is more or less what is meant by "having good reduction".

5.2.4. Meromorphic continuation. — Let us analyse the various terms of this formula. The stratum X_A has codimension $\mathrm{Card}(A)$ in X (or is empty); this implies easily that

$$q_v^{-\dim X}\,\mathrm{Card}(X_A(k_v)) = \mathrm{O}(q_v^{-\mathrm{Card}(A)}).$$

More precisely, since L_α is geometrically irreducible and X_α is an open subset of it, it follows from Deligne (1974) that

$$q_v^{-\dim X}\operatorname{Card}(X_\alpha(k_v)) = q_v^{-1} + \mathrm{O}(q_v^{-3/2}).$$

(In fact, the estimates of Lang & Weil (1954) suffice for that.) Similarly, $X_\emptyset = X \setminus L = U$, and

$$q_v^{-\dim X}\operatorname{Card}(X_\emptyset(k_v)) = 1 + \mathrm{O}(q_v^{-1/2}).$$

In fact, the vanishing assumption on the cohomology groups $\mathrm{H}^1(X, \mathscr{O}_X)$ and $\mathrm{H}^2(X, \mathscr{O}_X)$ implies that the remainder is even $\mathrm{O}(q_v^{-1})$. By an integration formula of Weil (1982), this expression is precisely equal to $\tau_{(X,L),v}(U(\mathfrak{o}_v))$.

We therefore obtain that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\operatorname{Re}(s-1) > -\delta$, then

$$Z_v(s) = \tau_{(X,L),v}(U(\mathfrak{o}_v)) \prod_{\alpha\in\mathscr{A}} (1 - q_v^{-1-d_\alpha(s-1)})^{-1} \left(1 + \mathrm{O}(q_v^{-1-\varepsilon})\right).$$

If we multiply this estimate by the convergence factors λ_v that we have introduced, and which satisfy $\lambda_v = 1 + \mathrm{O}(q_v^{-1})$, we obtain that the infinite product

$$\prod_{v \text{ finite}} \lambda_v Z_v(s) \prod_{\alpha\in\mathscr{A}} \zeta_{F,v}(1 + d_\alpha(s-1))^{-1}$$

converges absolutely and uniformly for $\operatorname{Re}(s-1) > -\delta/2$; it defines a holomorphic function $\Phi(s)$ on that half-plane. We have denoted by $\zeta_{F,v}$ the local factor at v of the Dedekind zeta function of the number field F. Then, the equality

$$\begin{aligned} Z(s) &= \frac{\mathrm{L}^*(1, M^0)}{\mathrm{L}^*(1, M^1)} \prod_{v\in\operatorname{Val}(F)} \lambda_v Z_v(s) \\ &= \frac{\mathrm{L}^*(1, M^0)}{\mathrm{L}^*(1, M^1)} \left(\prod_{\alpha\in\mathscr{A}} \zeta_F(1 + d_\alpha(s-1))\right) \Phi(s) \prod_{v \text{ archimedean}} Z_v(s) \end{aligned}$$

shows that Z converges absolutely, and defines a holomorphic function on $\{\operatorname{Re}(s) > 1\}$. Since the Dedekind zeta function ζ_F has a pole of order 1 at $s = 1$, $Z(s)$ admits a meromorphic continuation on the half-plane $\{\operatorname{Re}(s) > 1 - \delta/2\}$, whose only pole is at $s = 1$, with multiplicity $\operatorname{Card}(\mathscr{A})$.

5.2.5. The leading term. — Moreover,

$$\lim_{s\to 1}(s-1)^{\operatorname{Card}(\mathscr{A})} Z(s) = \frac{\mathrm{L}^*(1, M^0)}{\mathrm{L}^*(1, M^1)} \left(\prod_{\alpha\in\mathscr{A}} d_\alpha^{-1}\right) \zeta_F^*(1)^{\operatorname{Card}(\mathscr{A})} \Phi(1) \prod_{v \text{ archimedean}} Z_v(1).$$

By definition of Φ, one has

$$\begin{aligned} \Phi(1) &= \prod_{v \text{ finite}} \lambda_v \prod_{\alpha\in\mathscr{A}} \zeta_{F,v}(1)^{-1} Z_v(1) \\ &= \prod_{v \text{ finite}} \left(\lambda_v \prod_{\alpha\in\mathscr{A}} \zeta_{F,v}(1)^{-1}\right) \tau_{X,v}(X(F_v)). \end{aligned}$$

Comparing the Galois modules M^0 and M^1 for X and U, one can conclude that

$$\lim_{s\to 1}(s-1)^{\operatorname{Card}(\mathscr{A})} Z(s) = \tau_X(X(\mathbb{A}_F)) \prod_{\alpha\in\mathscr{A}} d_\alpha^{-1}. \tag{5.2.5.1}$$

5.2.6. Conclusion of the computation. — Finally, we have shown the following: The integral

$$s \mapsto \int_{U(\mathbb{A}_F)} H_L(x)^{-s}\,\mathrm{d}\tau_{(X,L)}(x)$$

converges for $\operatorname{Re}(s) > 1$, defines a holomorphic function on this half-plane. It has a meromorphic continuation on some half-plane $\{\operatorname{Re}(s) > 1-\delta\}$ (for some $\delta > 0$) whose only pole is at $s = 1$, has order $\operatorname{Card}(\mathscr{A})$ and its asymptotic behaviour at $s = 1$ is given by Equation (5.2.5.1).

5.3. Application to volume estimates

Let us finally derive our volume estimates from the analytic property of the function $Z(s)$. We are interested in the volume function defined by

$$V(B) = \tau_{(X,L)}\left(\{x \in U(\mathbb{A}_F)\,;\, H_L(x) \leqslant B\}\right)$$

for $B > 0$, more precisely in its asymptotic behaviour when $B \to \infty$.

Its Mellin-Stieltjes transform is given by

$$\int_0^\infty B^{-s}\mathrm{d}V(B) = \int_{U(\mathbb{A}_F)} H_L(x)^{-s}\,\mathrm{d}\tau_{(X,L)}(x) = Z(s).$$

By a slight generalization of Ikehara's Tauberian theorem, we conclude that $V(B)$ satisfies the following asymptotic expansion:

$$V(B) \sim \big((a-1)! \prod_{\alpha\in\mathscr{A}} d_\alpha\big)^{-1} \tau_X(X(\mathbb{A}_F)) B (\log B)^{a-1},$$

where $a = \operatorname{Card}(\mathscr{A})$ is the number of irreducible components of the divisor L.

Our argument, applied to different metrizations, also implies that when $B \to \infty$, the probability measure

$$\frac{1}{V(B)} \mathbf{1}_{H_L(x)\leqslant B} \mathrm{d}\tau_{(X,L)}(x)$$

on $U(\mathbb{A}_F)$ (viewed as a subset of $X(\mathbb{A}_F)$) converges to the measure

$$\frac{1}{\tau_X(X(\mathbb{A}_F))}\,\mathrm{d}\tau_X(x)$$

on $X(\mathbb{A}_F)$.

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